

Gödel's Theorem Fails for Π_1 Axiomatizations

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Abstract

We introduce a Π_1 set S for which Gödel's Second Incompleteness Theorem fails. In particular, we show $\text{ZF} \vdash \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \rightarrow \text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S))$. Then, we carefully analyze the relationship between $\text{Pf}_S(x)$ and $\text{Pf}_S(\text{Pf}_S(x))$ in order to show $\text{ZF} \vdash \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \rightarrow \exists x [\text{Pf}_S(x) \wedge \neg \text{Pf}_S(\text{Pf}_S(x))]$.

1 Preliminaries

Definition 1.1. Let \mathcal{G} denote the set of Gödel numbers for well-formed formulas of the first order system for Zermelo-Fraenkel Set Theory.

Definition 1.2. Let $\text{ZF} \subseteq \mathcal{G}$ denote the set of Gödel numbers for the standard axioms of Zermelo-Fraenkel Set Theory.

Definition 1.3. For all $A \subseteq \mathcal{G}$ and $\ulcorner X \urcorner \in \mathcal{G}$, let $\text{Pf}_A(\ulcorner X \urcorner)$ express that there is a proof of the well-formed formula represented by $\ulcorner X \urcorner$ from the set of formulas represented by members of A . For the remainder of the paper, we will omit the corner brackets and write $\text{Pf}_A(X)$ to improve readability.

We will take for granted that Pf can be recursively defined within the first order system for Zermelo-Fraenkel Set Theory. In addition, at the top level we write $\text{ZF} \vdash W$ to express that one could present a formal proof of the well-formed formula W using the first order system for Zermelo-Fraenkel Set Theory.

Definition 1.4. Let $\text{Con}(A)$ abbreviate $\neg \text{Pf}_A(0 = 1)$.

We will leave the following propositions as exercises for the reader.

Proposition 1.1. $\text{ZF} \vdash \forall A, B [A \subseteq B \rightarrow \forall x [\text{Pf}_A(x) \rightarrow \text{Pf}_B(x)]]$.

Proposition 1.2. $\text{ZF} \vdash \forall A, B [A \subseteq B \wedge \text{Con}(B) \rightarrow \text{Con}(A)]$.

Proposition 1.3. $\text{ZF} \vdash \forall A \forall x [\text{Con}(A) \wedge \text{Pf}_A(x) \rightarrow \text{Con}(A + x)]$.

Proposition 1.4. $\text{ZF} \vdash \forall A [\exists x \text{Pf}_A(\neg x \wedge x) \leftrightarrow \forall x \text{Pf}_A(x)]$.

In addition, we will make use of the following well-known theorems.

Deduction Theorem. $\text{ZF} \vdash \forall A \forall x, y [\text{Pf}_{A+x}(y) \leftrightarrow \text{Pf}_A(x \rightarrow y)]$.

Diagonal Lemma. *For every formula $p(x)$, there exists a sentence $\psi \in \mathcal{G}$ such that*

$$\text{ZF} \vdash \psi \leftrightarrow p(\ulcorner \psi \urcorner).$$

Gödel's Second Incompleteness Theorem. *Let a Σ_1 formula $\phi(x)$ be given. Let T denote the set associated with $\phi(x)$. If $\text{ZF} \vdash \text{ZF} \subseteq T \subseteq \mathcal{G}$, then*

$$\text{ZF} \vdash \text{Pf}_T(\text{Con}(T)) \rightarrow \neg \text{Con}(T).$$

The requirement on T being defined by a Σ_1 formula $\phi(x)$ is significant. It is necessary that $\phi(x)$ is embedded in the proof. See Appendix 4.1 for more details.

2 Gödel's Theorem Fails

Consider the following extension¹ of ZF:

$$S := \begin{cases} \text{ZF} + \text{Con}(\text{ZF}) & \text{if } \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \\ \text{ZF} & \text{otherwise.} \end{cases}$$

S is Π_1 because there is a program that enumerates the complement. This follows because ZF is decidable and we can determine if $\text{Con}(\text{ZF}) \notin S$ by searching for a proof of $0 = 1$ with axioms from $\text{ZF} + \text{Con}(\text{ZF})$.

If one could prove that $\text{Con}(\text{ZF})$ implies $\text{Con}(\text{ZF} + \text{Con}(\text{ZF}))$, then ZF proves its own inconsistency. However, we will prove in the following that $\text{Con}(\text{ZF})$ implies $\text{Con}(S)$.

¹Formally, one could define S using pairing, comprehension, and union.

Lemma 2.1. $ZF \vdash \text{Con}(S) \leftrightarrow \text{Con}(ZF)$.

Proof. The claim follows from the following three statements using the method of proof by cases.

- a) $ZF \vdash S = ZF + \text{Con}(ZF) \rightarrow [\text{Con}(S) \leftrightarrow \text{Con}(ZF)]$
- b) $ZF \vdash S = ZF \rightarrow [\text{Con}(S) \leftrightarrow \text{Con}(ZF)]$
- c) $ZF \vdash S = ZF + \text{Con}(ZF) \vee S = ZF$.

First, we show **a**.

$$S = ZF + \text{Con}(ZF) \Rightarrow \text{Con}(ZF + \text{Con}(ZF)) \quad (1)$$

$$\Rightarrow \text{Con}(S) \wedge \text{Con}(ZF) \quad (2)$$

$$\Rightarrow \text{Con}(S) \leftrightarrow \text{Con}(ZF). \quad (3)$$

(1) follows from the definition of S .

(2) follows from proposition 1.2 because $ZF \subseteq S \subseteq ZF + \text{Con}(ZF)$.

(3) follows from logical axioms.

Lastly, **b** follows from the axioms for equality and **c** follows from the definition of S and logical axioms. \square

Theorem 2.1. $ZF \vdash \text{Con}(ZF + \text{Con}(ZF)) \rightarrow \text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S))$.

Proof. First, by proposition 1.2, we have $ZF \vdash \text{Con}(ZF + \text{Con}(ZF)) \rightarrow \text{Con}(S)$ because $S \subseteq ZF + \text{Con}(ZF)$. Next, we show $ZF \vdash \text{Con}(ZF + \text{Con}(ZF)) \rightarrow \text{Pf}_S(\text{Con}(S))$.

$$\text{Con}(ZF + \text{Con}(ZF)) \Rightarrow S = ZF + \text{Con}(ZF) \quad (4)$$

$$\Rightarrow \text{Pf}_S(\text{Con}(ZF)) \quad (5)$$

$$\Rightarrow \text{Pf}_S(\text{Con}(S)). \quad (6)$$

(4) follows from the definition of S .

(5) follows because $\text{Con}(ZF) \in S$.

(6) One can use the proof of lemma 2.1 to show $ZF \vdash \text{Pf}_{ZF}(\text{Con}(S) \leftrightarrow \text{Con}(ZF))$. Since $ZF \subseteq S$, we can apply proposition 1.1 to get $ZF \vdash \text{Pf}_S(\text{Con}(S) \leftrightarrow \text{Con}(ZF))$. \square

3 Incompatible Proof Levels

We introduced a set S whose members depend on a property that is potentially independent of ZF. In particular, $\text{Con}(\text{ZF}) \in S$ if and only if $\text{Con}(\text{ZF} + \text{Con}(\text{ZF}))$. We will show that if $\text{Con}(\text{ZF}) \in S$, then S has incompatible proof levels, that is S proves the sentence $\text{Con}(S)$, but does not prove that it proves $\text{Con}(S)$.

Lemma 3.1. $\text{ZF} \vdash \text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S)) \rightarrow \text{Con}(\text{ZF} + \text{Con}(\text{ZF}))$.

Proof.

$$\text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S)) \Rightarrow \text{Con}(S) \wedge \text{Pf}_S(\text{Con}(\text{ZF})) \quad (7)$$

$$\Rightarrow \text{Con}(S + \text{Con}(\text{ZF})) \quad (8)$$

$$\Rightarrow \text{Con}(\text{ZF} + \text{Con}(\text{ZF})). \quad (9)$$

(7) One can use the proof of lemma 2.1 to show $\text{ZF} \vdash \text{Pf}_{\text{ZF}}(\text{Con}(S) \leftrightarrow \text{Con}(\text{ZF}))$. Since $\text{ZF} \subseteq S$, we can apply proposition 1.1 to get $\text{ZF} \vdash \text{Pf}_S(\text{Con}(S) \leftrightarrow \text{Con}(\text{ZF}))$.

(8) follows from proposition 1.3.

(9) follows because $S + \text{Con}(\text{ZF}) = \text{ZF} + \text{Con}(\text{ZF})$. □

Theorem 3.1. $\text{ZF} \vdash \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \rightarrow \neg \text{Pf}_S(\text{Pf}_S(\text{Con}(S)))$.

Proof.

$$\text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \Rightarrow \neg \text{Pf}_{\text{ZF} + \text{Con}(\text{ZF})}(\text{Con}(\text{ZF} + \text{Con}(\text{ZF}))) \quad (10)$$

$$\Rightarrow \neg \text{Pf}_S(\text{Con}(\text{ZF} + \text{Con}(\text{ZF}))) \quad (11)$$

$$\Rightarrow \neg \text{Pf}_S(\text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S))) \quad (12)$$

$$\Rightarrow \neg \text{Pf}_S(\text{Pf}_S(\text{Con}(S))). \quad (13)$$

(10) follows from Gödel's Second Incompleteness Theorem.

(11) follows from proposition 1.1 because $S \subseteq \text{ZF} + \text{Con}(\text{ZF})$.

(12) One can use the proof of lemma 3.1 to show

$$\text{ZF} \vdash \text{Pf}_{\text{ZF}}(\text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S))) \rightarrow \text{Con}(\text{ZF} + \text{Con}(\text{ZF})).$$

Since $\text{ZF} \subseteq S$, we can apply proposition 1.1 to get

$$\text{ZF} \vdash \text{Pf}_S(\text{Con}(S) \wedge \text{Pf}_S(\text{Con}(S))) \rightarrow \text{Con}(\text{ZF} + \text{Con}(\text{ZF})).$$

(13) follows from theorem 2.1 because $\text{ZF} \vdash \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \rightarrow \text{Pf}_S(\text{Con}(S))$. \square

Corollary 3.1. $\text{ZF} \vdash \text{Con}(\text{ZF} + \text{Con}(\text{ZF})) \rightarrow \exists x [\text{Pf}_S(x) \wedge \neg \text{Pf}_S(\text{Pf}_S(x))]$.

Corollary 3.2. *ZF proves that the following are equivalent:*

- (1) $\neg \text{Con}(\text{ZF} + \text{Con}(\text{ZF}))$
- (2) $\text{Pf}_{\text{ZF}}(\neg \text{Con}(\text{ZF}))$
- (3) $\text{Pf}_S(\neg \text{Con}(S))$
- (4) $\text{Pf}_S(\text{Pf}_S(\text{Con}(S)))$.

4 Appendix

4.1 Gödel's Second Incompleteness Theorem

Gödel's Theorem is a theorem scheme. If a Σ_1 formula $\phi(x)$ is provided, then one could carry out the proof. The assumption that $\phi(x)$ is Σ_1 is significant. Since $\phi(x)$ is Σ_1 , T is computably enumerable and one could write a program p that enumerates codings of T -proofs *i.e.* proofs whose axioms are from T . Therefore, if t is a coding of a T -proof, then there is a computation for p that accepts t . The existence of a computation implies the existence of a proof that t is in fact a T -proof. Since t is an arbitrary T -proof, one could formalize the preceding to get

$$\text{ZF} \vdash \forall x [\text{Pf}_T(x) \rightarrow \text{Pf}_T(\text{Pf}_T(x))]$$

which is needed to carry out the proof that T proves its own consistency implies T is inconsistent.

There is a terrible subtlety in the preceding discussion. We require that a Σ_1 formula $\phi(x)$ is provided. In particular, we cannot generalize over all Σ_1 sets T . Pick two distinct Σ_1 sets Y_1 and Y_2 . Consider the following set

$$W := \begin{cases} Y_1 & \text{if CH} \\ Y_2 & \text{otherwise.} \end{cases}$$

If CH denotes the Continuum Hypothesis, then whether W is associated with Y_1 's formula or Y_2 's formula is independent of ZF. Therefore, we cannot provide a Σ_1 formula

for W . Formally, Gödel’s Theorem will not apply to W because we need to use the Σ_1 formula to prove “that t is in fact a T -proof”, as stated above.

4.2 Known Results for Complete Theories

It is worth noting that complete extensions of ZF are known to have properties similar to S . In particular, if one defines a complete extension T of ZF, then we observe that $\text{ZF} \vdash \text{Con}(T + \text{Con}(T)) \rightarrow \text{Con}(T) \wedge \text{Pf}_T(\text{Con}(T))$ and using the Diagonal Lemma can show $\text{ZF} \vdash \text{Con}(T + \text{Con}(T)) \rightarrow \exists x [\text{Pf}_T(x) \wedge \neg \text{Pf}_T(\text{Pf}_T(x))]$.

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